Quantum fluctuations of a vortex in a dilute Bose-Einstein condensate

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A vortex in a quasi two-dimensional Bose-Einstein condensate is subject to the Magnus force and can be effectively described as a planar particle in a uniform magnetic field. Quantization of this effective particle leads to the lowest Landau level where the most localized wave function is a gaussian. In this gaussian state vortex position seems to fluctuate with an average magnitude set by the magnetic width of the gaussian. We readdress this problem using the number-conserving version of the Bogoliubov theory. We find that the Bogoliubov mode that might be interpreted as a fluctuation of vortex position actually does not contribute to position fluctuation at all. The only non-zero contribution comes from phonons but it is an order of magnitude less than the simple estimate, based on the magnetic width of the effective gaussian wave packet.

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I. INTRODUCTION

Bose-Einstein condensation of dilute alkali gases [1] opens up new possibilities to investigate Bose-Einstein-condensed systems under flexible and well controlled conditions. For example it is possible to manipulate with the dimensionality of the trap to achieve quasi one or two-dimensional condensates [2]. Temperatures can be reduced to the pK range [3] bringing the system close to the quantum regime. Various manifestations of superfluidity have been subject of intensive research, the most spectacular have been probably vortices [4] and Abrikosov vortex lattices [5] in rotating traps. It is predicted [6] that further increase in the rotation speed of a quasi-two-dimensional trap can drive the system from the regime of BEC to the regime of strongly correlated bosonic quantum Hall liquids with even denominator filling fractions.

A dilute bosonic gas in a rotating two-dimensional trap at very low temperature is a setup required for quantum Hall liquids but at the same time it can be used to experimentally test quantum fluctuations of a 2D vortex in a dilute Bose-Einstein condensate. In fact melting of the Abrikosov vortex lattice driven by quantum fluctuations of vortices is the first step from a Bose-Einstein condensate towards strongly correlated quantum Hall liquids. In this paper we develop a theory of quantum vortex fluctuations.

II. VORTEX FLUCTUATIONS

An effective action for a planar vortex has been worked out in Ref. [7]. In terms of vortex position on the plane $\vec{X} = (\mathcal{X}, \mathcal{Y})$ the action reads

$$\mathcal{S}_{\rm eff} \;=\; \int dt\, \left[\frac{m_{\rm eff}}{2} \dot{\vec{X}}^2 + \frac{\kappa}{2} \hbar \rho \; \hat{e}_z \left(\vec{X} \times \dot{\vec{X}} \right) \right] \; . \tag{1}$$

Here $m_{\rm eff}$ is an effective mass of the vortex that depends on the size of the system, ρ is planar density of atoms in a uniform condensate, the integer κ is the winding number of the vortex, and $\hat{e}_z$ is a unit vector along the z-axis normal to the plane. Formally this is the action of a planar particle in a uniform magnetic field normal to the plane, as we can see from its Euler-Lagrange equation: $m_{\rm eff}\ddot{\vec{X}} = \kappa\hbar\rho\; \dot{\vec{X}} \times \hat{e}_z$. The effective Lorenz force on the right hand side is the Magnus force. Because vortices with $|\kappa| > 1$ are unstable, in the following we consider only $\kappa = +1$.

The effective particle can be quantized and its eigenstates can be classified in the degenerate Landau levels. The energy gap between Landau levels is $\frac{\hbar\rho}{m_{\rm eff}}$ so in the limit of vanishing effective mass the spectrum can be truncated to the lowest Landau level (LLL). The degenerate eigenmodes of the LLL can be chosen as $(\mathcal{Z}^*)^l \; e^{-\frac{1}{2}\rho \mathcal{Z}^* \mathcal{Z}}$. Here $\mathcal{Z} = \mathcal{X} + i\; \mathcal{Y}$ is a complex position of the vortex. We get the most localized eigenstate when the non-negative integer $l = 0$:

$$e^{-\frac{1}{2}\rho \mathcal{Z}^* \mathcal{Z}} = e^{-\frac{1}{2}\rho (\mathcal{X}^2 + \mathcal{Y}^2)} \; . \tag{2}$$

This gaussian seems to determine the quantum fluctuation of the vortex position.

In the next Section we rederive this result in the framework of the particle-number-conserving Bogoliubov theory [8], but then we use the same Bogoliubov theory to show that this result is incomplete and even misleading.

III. VORTEX FLUCTUATIONS IN THE BOGOLIUBOV THEORY

In the mean-field approximation a stationary condensate wave function is a solution of the time-independent Gross-Pitaevskii equation

$$\mu\phi = -\frac{\hbar^2}{2m} \nabla^2 \phi + N g_0 |\phi|^2 \phi + V\phi \; . \tag{3}$$

Here g_0 is an effective 2D interaction strength and N is a total, conserved number of atoms in the box. In this section we ignore the trap potential V and consider a uniform 2D condensate in a box. In the strong interaction Thomas-Fermi (TF) regime ($Ng_0 \equiv g \gg 1$) the uniform condensate has a wave function $\phi = \sqrt{\rho/N}$ with $\mu = \rho g_0$. In this case vortex has a wave function $\phi_0(r,\theta) = f(r)\exp(i\theta)$ in polar coordinates r and θ, where the modulus $f(r)$ interpolates between $f(0) = 0$ and $f(\infty) = \sqrt{\rho/N}$.

Small fluctuations around the vortex can be expanded into Bogoliubov modes with complex amplitudes b_m

$$\phi_b(\vec{x}) = \mathcal{N}\left[\phi_0(\vec{x}) + \sqrt{\frac{2}{N}}\sum_m b_m u_m(\vec{x}) + b_m^* v_m^*(\vec{x})\right] \ , \tag{4}$$

where

$$\begin{aligned}\mathcal{N}^{-1}(b,b^*) \quad &= \\ 1 + \frac{2}{N}\textstyle\sum_{m,n} & b_m^* b_n\langle u_m|u_n\rangle + b_m b_n\langle v_m^*|u_n\rangle + \\ & b_m^* b_n^*\langle u_m|v_n^*\rangle + b_m b_n^*\langle v_n|v_m\rangle \end{aligned} \tag{5}$$

is the normalization factor, such that $\langle\phi_b|\phi_b\rangle = 1$. In addition to the usual phonon modes with the characteristic phonon dispersion relation, there is a normalizable mode with zero frequency

$$\begin{aligned} u_0 = -\sqrt{\frac{N}{\rho}}\,\frac{\partial}{\partial z}\phi_0 \; &= \; -\sqrt{\frac{N}{\rho}}\,\left(\frac{df}{dr} + \frac{f}{r}\right) \ , \\ v_0^* = -\sqrt{\frac{N}{\rho}}\,\frac{\partial}{\partial z^*}\phi_0 \; &= \; -\sqrt{\frac{N}{\rho}}\,\left(\frac{df}{dr} - \frac{f}{r}\right)e^{2i\theta} \ . \end{aligned} \tag{6}$$

Here $z = x + iy = re^{i\theta}$. This mode is normalized in the usual way $\langle u_0|u_0\rangle - \langle v_0|v_0\rangle = 1$. When truncated only to the zero mode the Bogoliubov expansion (4) becomes

$$\phi_b \approx \phi_0 - \mathcal{Z}\frac{\partial}{\partial z}\phi_0 - \mathcal{Z}^*\frac{\partial}{\partial z^*}\phi_0 \ . \tag{7}$$

This expression can be interpreted as the leading terms of a Taylor expansion in powers of the complex vortex position $\mathcal{Z} = \mathcal{X} + i\mathcal{Y} = \sqrt{\frac{2}{\rho}}\ b_0$. To see quantum fluctuations of $\mathcal{Z}$ we must go to the quantum version of the N-conserving Bogoliubov theory.

Any N-particle state $|\psi\rangle$ can be written as a quantum superposition over condensates with different condensate wave functions ϕ_b

$$|\psi\rangle \;=\; \int d^2b\ \psi(b,b^*)\ |N:\phi_b\rangle \ . \tag{8}$$

Here d^2b is an abbreviation for $\prod_m d^2b_m$. The state $|N:\phi_b\rangle$ is a Fock state with N atoms in the wave function ϕ_b from Eq.(4). For $N \gg 1$ the action of the quasiparticle annihilation operator

$$\hat{b}_m \;=\; \frac{\hat{a}_0^\dagger\hat{u}_m - \hat{a}_0\hat{v}_m^\dagger}{\sqrt{N}} \ , \tag{9}$$

with $\hat{u}_m = \langle u_m|\hat{\psi}\rangle$ and $\hat{v}_m = \langle v_m^*|\hat{\psi}\rangle$, on the state $|\psi\rangle$, turns out to be very simple:

$$\begin{aligned} &\hat{b}_m\ |\psi\rangle \;= \\ &\int d^2b\ \psi(b,b^*)\ \hat{b}_m\ |N:\phi_b\rangle \ \overset{N\gg 1}{\approx} \\ &\int d^2b\ \left[\frac{1}{\sqrt{2}}\left(b_m + \frac{\partial}{\partial b_m^*}\right)\psi(b,b^*)\right]\ |N:\phi_b\rangle \ . \end{aligned} \tag{10}$$

The annihilation (creation) operator $\hat{b}_m^{(\dagger)}$ acts on a wave function $\psi(b,b^*)$ as a differential operator. A 'b-representation' of the operators is

$$\hat{b}_m \;=\; \frac{1}{\sqrt{2}}\left(b_m + \frac{\partial}{\partial b_m^*}\right) \ , \tag{11}$$

$$\hat{b}_m^\dagger \;=\; \frac{1}{\sqrt{2}}\left(b_m^* - \frac{\partial}{\partial b_m}\right) \ . \tag{12}$$

It is easy to check, that $[\hat{b}_m, \hat{b}_n^\dagger] = \delta_{mn}$.

The Bogoliubov vacuum state $|0_b\rangle$ is the state without any Bogoliubov quasiparticles, so its wave function is annihilated by all the operators $\hat{b}_m$:

$$\hat{b}_m|0_b\rangle = 0, \tag{13}$$

it implies:

$$\psi_0(b,b^*) \;\sim\; \prod_m e^{-b_m^* b_m} \;\equiv\; e^{-b^* b} \ . \tag{14}$$

If $b_m^\dagger$ is applied l_m times to the vacuum state in order to create l_m quasiparticles, then one gets the l_m-quasiparticle wave function $(\sqrt{2}b_m^*)^{l_m}\ e^{-b^* b}$. It is quite remarkable that these wave functions look just like the orbitals of the lowest Landau level. This may be a little bit surprising because after all, they are the eigenstates of the quasiparticle harmonic oscillators and not of any particles in a magnetic field. Also, more generally, a quasiparticle Fock state with definite numbers of different quasiparticles has a characteristic 'polynomial $\times$ exponential' form $\left(\prod_m\ (b_m^*)^{l_m}\right)\ e^{-b^* b}$. A general wave function is $\psi(b,b^*) = P(b^*)e^{-b^* b}$ with an arbitrary polynomial in b^*'s.

As we already could see in Eq.(7), in the case of our planar vortex one of the fictitious particle coordinates b_0 materializes as a complex vortex position $\mathcal{Z} = \sqrt{\frac{2}{\rho}}b_0$. The Bogoliubov vacuum wave function (14) for this mode becomes

$$e^{-\frac{1}{2}\rho\mathcal{Z}^*\mathcal{Z}} \tag{15}$$

and it is identical with the wave function (2). In contrast to Eq.(2) the wave function (15) is derived from first principles without any reference to the effective action (1).

In the following we use the same first principles to show that Eqs.(2) and (15) are misleading. It does not make sense to truncate the theory to the vortex zero mode because this mode is not independent from other Bogoliubov modes.

IV. A CONTRADICTION

The quantum fluctuations described by Eq.(15) result in quantum depletion of atoms from the condensate wave function $\phi_0(\vec{x})$. The density of atoms depleted from the condensate in the state (8) with the wave function (15) truncated to the vortex mode can be worked out as

$$d\rho_0(\vec{x}) \ \stackrel{?}{=} \ |u_0(\vec{x})|^2 + |v_0(\vec{x})|^2 \ . \qquad (16)$$

Here u_0 and v_0 come in a symmetric way like in Eq.(4). This $d\rho_0(\vec{x}) \neq 0$ at $r = 0$, the non-zero contribution is coming from the term $|u_0(\vec{x})|^2$. The fluctuating vortex position results in a non-zero density of depleted atoms inside vortex core. This is quite natural: a vortex with a fixed position has an empty core but when the position is uncertain then the average density of atoms inside the core in non-zero. There is a one to one correspondence between the magnitude of position fluctuations and the density of depleted atoms in the core.

On second thought, Eq.(16) appears strange. After all the density of depleted atoms in $|0_b\rangle$ state, using the Bogoliubov theory, is given by the sum $\sum_{m=0}^{\infty} |v_m(\vec{x})|^2$ which contains all v_m's but no u_m's. When truncated only to the vortex mode the sum becomes

$$d\rho_0(\vec{x}) \ \stackrel{?}{=} \ |v_0(\vec{x})|^2 \qquad (17)$$

in apparent contradiction with Eq.(16). Both Eq.(16) and Eq.(17) result from a truncation of the Bogoliubov theory to the vortex mode. However, the two truncations are done in a different way. The resulting contradiction shows that such a truncation does not make sense.

The reason why it makes no sense is that different Bogoliubov modes (u_m, v_m) are not independent from each other. Their wave functions are not mutually orthogonal in the usual sense: for example, in general $\langle u_m | u_n \rangle \neq 0$ for $m \neq n$. The wave function of the vacuum (14) factorizes implying independence of different b_m's, but the nontrivial correlations between different Bogoliubov modes are hidden in the N-particle condensate $|N : \phi_b\rangle$ making the mode independence rather elusive.

V. PHONONS DRIVE VORTEX FLUCTUATIONS

In contrast to Eq.(16) equation (17) shows vanishing density of depleted atoms at $r = 0$, i.e. there are no vortex position fluctuations originating from the vortex mode. At the same time the contradiction between Eqs.(16) and (17) shows that we cannot judge on position fluctuations using any of the two truncations, but we have to use the full expression

$$d\rho(r = 0) \ = \ \sum_{m=0}^{\infty} |v_m(r = 0)|^2 \qquad (18)$$

including all Bogoliubov modes. We know that the vortex mode ($m = 0$) has no contribution to $d\rho(r = 0)$. To work out the contribution from the other modes in a controlled way we put the system in a 2D harmonic trap with a frequency ω. In the dimensionless trap units the stationary Gross-Pitaevskii equation becomes

$$\mu\phi = -\frac{1}{2}\nabla^2\phi + \frac{1}{2}r^2\phi + g|\phi|^2\phi \ . \qquad (19)$$

Its vortex solutions $\phi_0 = f(r)e^{i\theta}$ for several different values of g are shown in trap units in Fig.1.

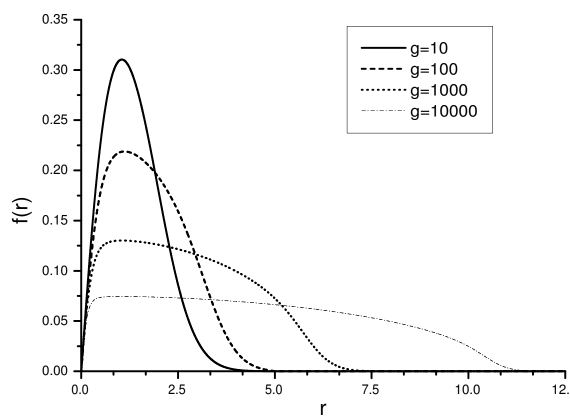

FIG. 1. Profiles of the vortex wave function for several values of the interaction strength g. The solutions were obtained by relaxation to the ground state with vortex symmetry. Except for the vortex core at $r \approx 0$, the profiles $f(r)$ for $g = 1000, 10000$ match well with the Thomas-Fermi profile $f_{\rm TF}(r) = \sqrt{\frac{R^2 - r^2}{2g}}$. Here $R = (4g/\pi)^{1/4}$ is the Thomas-Fermi radius in a two dimensional trap.

The Bogoliubov modes are solutions of the Bogoliubov-de Gennes equations

$$\begin{aligned} \omega_m u_m &= \mathcal{H}_{GP} u_m + g\phi_0^2 v_m \ , \\ -\omega_m v_m &= \mathcal{H}_{GP} v_m + g(\phi_0^*)^2 u_m \ . \end{aligned} \qquad (20)$$

Here $\mathcal{H}_{GP} = -\frac{1}{2}\nabla^2 + \frac{1}{2}r^2 + 2g|\phi_0|^2 - \mu$. The symmetry of $\phi_0 \sim e^{i\theta}$ implies that

$$u(r,\theta) \;=\; U(r)\; e^{il\theta+2i\theta}\;,$$
$$v(r,\theta) \;=\; V(r)\; e^{il\theta}\;. \tag{21}$$

The Bogoliubov modes can be classified into different multiplets with definite angular momentum l. As only the modes with $l=0$ can contribute to $d\rho(r=0)$ in Eq.(18), from now on we focus only on the $l=0$ multiplet.

The Bogoliubov-de Gennes equations were numerically solved for $l=0$ using the vortex solutions ϕ_0 (Fig.1). In the numerical calculations the envelopes $U(r)$ and $V(r)$ were expressed as finite linear combinations of the 2D harmonic oscillator eigenfunctions up to certain energy cut-off. All the numerical quantities presented below were checked for convergence with the increasing energy cut-off. We also tested our numerical code in the limit of $g \ll 1$ where some perturbative analytical results can be compared with numerics. The Bogoliubov modes of the $l=0$ multiplet can be classified as follows.

- The anomalous mode which is a harmonic trap counterpart of the vortex zero mode (6). In the trap the mode acquires a non-zero negative frequency $\omega_a < 0$, where $\omega_a \to 0$ in the limit of $g \to \infty$ [9]. It was tested in [10].

- The collective oscillations mode with the trap frequency $\omega_1 = 1$,

$$u_1 \;=\; \left(\frac{df}{dr} - \frac{f}{r} - rf\right) e^{2i\theta}\;,$$
$$v_1 \;=\; \left(\frac{df}{dr} + \frac{f}{r} + rf\right)\;. \tag{22}$$

The frequency ω_1 of the numerical counterpart of this exact mode was kept within 0.01 of the exact $\omega_1 = 1$. Contribution of the trap mode to the density of depleted atoms in the core can be immediately estimated as

$$|v_1(r=0)|^2 \;=\; \left|\frac{df}{dr}(r=0)\right|^2 \;\approx\; \frac{1}{R\xi} \;=\; \mathcal{O}(g^0)\;. \tag{23}$$

This contribution does not depend on g. For large g it is negligible as compared to phonon modes (see next point). Even for a finite g this trap mode is a collective excitation of the center of mass of the atomic cloud which has nothing to do with the motion of the vortex with respect to the condensate. For these two reasons we do not include the trap mode in $d\rho(r=0)$ even for finite g.

- Phonon modes with positive frequencies $\omega_{m>1} > 0$. These modes give a nonzero contribution to $d\rho(r=0)$ which grows with g as $\sqrt{g}$. The phonon contribution dominates for large g and it results in genuine fluctuation of the vortex position with respect to the condensate. Rather surprisingly, fluctuation of vortex position, if any, turns out to come from phonons and not from the vortex mode.

We compared the dominating phonon contribution $d\rho_{\rm phonons}(r=0) = \sum_{m\geq 2} |v_m(r=0)|^2$ to the quantum depletion in a uniform condensate without any vortex. If the density of the uniform condensate is equal to the density of our condensate near the center of the trap, then the quantum depletion in the uniform condensate is $d\rho_{\rm uniform} = g^{1/2}/4\pi^{3/2}$. We compare our numerical value of $d\rho_{\rm phonons}(r=0)$ to this uniform depletion in Fig.2 where we plot the reduction factor

$$\varepsilon \;=\; \frac{d\rho_{\rm phonons}(r=0)}{d\rho_{\rm uniform}}\;. \tag{24}$$

For large g the reduction factor ε stabilizes at approximately 0.23. We conclude that the density of depleted atoms inside the vortex core is a factor of 0.23 less than the same depletion outside the vortex. Apparently the centrifugal force due to vortex rotation is quite efficient in throwing away from the core not only the condensate but also the non-condensed atoms.

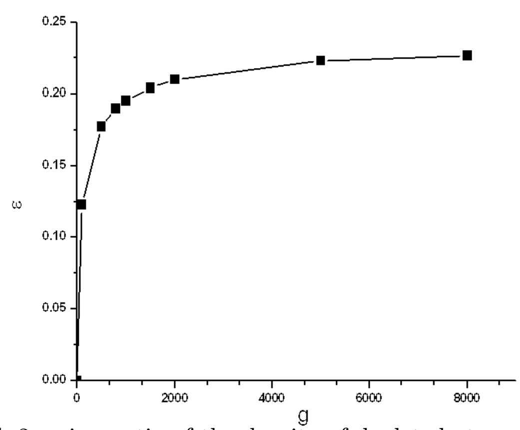

FIG. 2. ε is a ratio of the density of depleted atoms inside the vortex core at $r=0$ to the density of depleted atoms in a uniform condensate without any vortex (or depletion outside the vortex core). Small ε means that the non-condensed atoms are quite efficiently expelled from the core by the centrifugal force. ε stabilizes around 0.23 for large g when the Thomas-Fermi condensate can be very well approximated by a uniform condensate, at least locally close to the vortex in the center of the trap. $\varepsilon \approx 0.23$ is the asymptotic value of the reduction factor in a uniform condensate.

To summarize we find that for a large g the density of depleted atoms in the core is dominated by the phonon contribution and is approximately given by

$$d\rho(r=0) \;\approx\; 0.23\;\frac{g^{1/2}}{4\pi^{3/2}}\;. \tag{25}$$

It is time to compare this phonon contribution with the simple estimate (16) based on the truncation to the vortex mode. In the Thomas-Fermi limit we estimate $|u_0(r=0)|^2 = \frac{1}{4\pi}\frac{N}{\rho}\left|\frac{df}{dr}(r=0)\right|^2 \approx 0.090 g^{1/2}$. The ratio of the true to the naive density reads

$$\delta \;=\; \frac{d\rho(r=0)}{|u_0(r=0)|^2} \;\approx\; 0.11 \; . \tag{26}$$

We conclude that in the Thomas-Fermi limit the actual fluctuations of the vortex position are an order of magnitude less than the simple estimate based on quantization of the effective action or equivalent truncation of the Bogoliubov theory to the vortex mode.

VI. SUMMARY

This paper has the following logic. We start from an effective action for vortex dynamics in two dimensions, quantize it and obtain an effective wave function for the position of the vortex. Dispersion of this gaussian wave function seems to estimate quantum fluctuations in vortex position. Next we introduce the number conserving Bogoliubov theory and rederive the same wave function. This establishes a connection between the effective action quantization and the Bogoliubov theory: the former gives the same result as the latter when the latter is truncated to the vortex mode. Next, in the framework of the Bogoliubov theory we show that the truncation is not unique and that vortex position fluctuations critically depend on the way of truncating to the vortex mode. It turns out that the vortex mode in the Bogoliubov theory is not independent from the phonon modes so that all modes have to be included in order to correctly estimate quantum fluctuations. Rather surprisingly, in the full untruncated Bogoliubov theory quantum fluctuations of vortex position come from phonon modes and not from the vortex mode. What is more these phonon-induced quantum fluctuations are an order of magnitude less than the simple estimate based on the truncation to the vortex mode or on the quantization of the effective action. We want to stress this result means only that the naive quantization of the effective action (1) does not work properly, not that the effective action is incorrect.

We also found that the density of depleted atoms inside a vortex core is a factor of 0.23 less than outside the core.

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